on $(0,1)$ $$-e^x \frac{d^2u}{dx^2} - e^x \frac{du}{dx} = \tan(x)$$

$$\frac{du}{dx}(1) + 10\,u(1) = -5 \qquad u(0) = 1$$

1. • make a homogeneous boundary condition

$u(0)=1 \Rightarrow$ define $\bar{u}(x) = 1$, $\tilde{u}(x) = u(x) - \bar{u}(x)$, then $\tilde{u}(0) = 0$

$\Rightarrow u(x) = \tilde{u}(x) + 1$

• other boundary condition becomes

$$\frac{du}{dx}(1) + 10\,u(1) = \frac{d\tilde{u}}{dx}(1) + 10\,\tilde{u}(1) + 10 = -5$$

1

$$\Rightarrow \frac{d\tilde{u}}{dx}(1) + 10\,\tilde{u}(1) = -15$$

• Hence, boundary value problem for $\tilde{u}(x)$

$$-e^x \frac{d^2\tilde{u}}{dx^2} - e^x \frac{d\tilde{u}}{dx} = \tan(x)$$

$$\frac{d\tilde{u}}{dx}(1) + 10\,\tilde{u}(1) = -15 \qquad \tilde{u}(0) = 0$$

• weak Galerkin form:

residuals $r_1(\tilde{u}) = \tan x - \left(-e^x \frac{d^2\tilde{u}}{dx^2} - e^x \frac{d\tilde{u}}{dx}\right)$

$r_2(\tilde{u}) = -15 - \left(\frac{d\tilde{u}}{dx}(1) + 10\,\tilde{u}(1)\right)$

find $u_\nu \in \mathcal{V}$ s.t. $(v, r_1(u_\nu)) + \alpha\, v(1)\, r_2(u_\nu) = 0 \quad \forall v \in \mathcal{V}$

1

i.e. $\left(v, -e^x \frac{d^2u_\nu}{dx^2} - e^x \frac{du_\nu}{dx}\right) + \alpha\left(\frac{du_\nu}{dx}(1) + 10\,u_\nu(1)\right) v(1)$

$= (v, \tan x) - 15\alpha\, v(1) \quad \forall v \in \mathcal{V}$

bilinear form $a(v,u) = \left(v, -e^x \frac{d^2u}{dx^2} - e^x \frac{du}{dx}\right) + \alpha\left(\frac{du}{dx}(1) + 10u(1)\right) \cdot v(1)$

linear form $F(v) = (v, \tan x) - 15\alpha\, v(1)$

0.5 function space $\mathcal{V} = \{ u \in H^1(0,1) \mid u(0) = 0 \}$

• bi-linear form

$$\left(v, -e^x \frac{d^2u}{dx^2} - e^x \frac{du}{dx}\right) = \int_0^1 v\left(-e^x \frac{d^2u}{dx^2} - e^x \frac{du}{dx}\right) dx$$

1

$$= \int_0^1 \underbrace{-ve^x}_{f} \underbrace{\frac{d}{dx}\left(\frac{du}{dx} + u\right)}_{g'} dx$$

$$= -ve^x\left(\frac{du}{dx}+u\right)\Big|_0^1 - \int_0^1 -\left(\frac{dv}{dx}e^x + e^x v\right)\left(\frac{du}{dx}+u\right)dx$$

p.i.

$$\underset{v(0)=0}{=} -v(1)e\left(\frac{du}{dx}(1)+u(1)\right) + \left(e^x\frac{dv}{dx} + e^x v, \frac{du}{dx}+u\right)$$

bi-linear form with only first-order derivatives

$$\Rightarrow a(v,u) = \left(e^x\frac{dv}{dx} + e^x v, \frac{du}{dx}+u\right) - v(1)(e-\alpha)\frac{du}{dx}(1) - v(1)(e-10\alpha)u(1)$$

2. find α s.t $a(u,u) \geq 0$

$$a(u,u) = \left(e^x\frac{du}{dx} + e^x u, \frac{du}{dx}+u\right) - u(1)(e-\alpha)\frac{du}{dx}(1) - u(1)(e-10\alpha)u(1)$$

$$\left(e^x\frac{du}{dx} + e^x u, \frac{du}{dx}+u\right) = \left(e^x\left(\frac{du}{dx}+u\right), \frac{du}{dx}+u\right)$$

$$= \int_0^1 \underbrace{e^x}_{>0 \text{ on } (0,1)} \underbrace{\left(\frac{du}{dx}+u\right)^2}_{\geq 0} dx \geq 0$$

$$-u(1)(e-10\alpha)u(1) = (-e+10\alpha)u(1)^2 \overset{\text{with } \alpha=e}{=} 9e\,u(1)^2 > 0$$

$-u(1)(e-\alpha)\frac{du}{dx}(1)$ sign undetermined $\Rightarrow$ take $\alpha = e$

$\Rightarrow$ with $\alpha = e$ is the bi-linear form non negative

3. if $a(u,v)$ is coercive and symmetric an associated minimization problem is there

- coercivity is assumed
- with $\alpha = e$

$$a(v,u) = \left(e^x\left(\frac{dv}{dx}+v\right), \frac{du}{dx}+u\right) + 9e\,v(1)u(1)$$

$$= \int_0^1 e^x\left(\frac{dv}{dx}+v\right)\left(\frac{du}{dx}+u\right)dx + 9e\,u(1)v(1)$$

$$= \int_0^1 e^x\left(\frac{du}{dx}+u\right)\left(\frac{dv}{dx}+v\right)dx + 9e\,u(1)v(1)$$

$$= \left(e^x\left(\frac{du}{dx}+u\right), \frac{dv}{dx}+v\right) + 9e\,u(1)v(1)$$

$$= a(u,v) \Rightarrow a(\cdot,\cdot) \text{ symmetric}$$

$\Rightarrow$ a minimization problem is associated

4. V_h piecewise linear interpolation polynomials
interpolation points $x_i = ih$ $h = \frac{1}{n}$
$v_h(0) = 0$ for all $v_h \in V_h$

- Linear basis $\{\phi_1, \dots, \phi_n\}$ for V_h

1.5

$i=1,\dots,n$ $\phi_i(x_j) = \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$ $j = 0 \dots n$

$\phi_i(x)\big|_{[x_j, x_{j+1}]}$ linear $j = 0,\dots,n-1$

$$\Rightarrow \phi_i(x) = \begin{cases} 0 & x \geq x_{i+1} \\ & x \leq x_{i-1} \\ \frac{x - x_{i-1}}{h} & x_{i-1} < x \leq x_i \\ \frac{x_{i+1} - x}{h} & x_i < x < x_{i+1} \end{cases}$$

- $u(x) \approx \sum_{i=1}^{n} c_i \phi_i(x)$ $A\vec{c} = b$

0.5 $$A_{ii} = a(\phi_i, \phi_i) = \int_{x_{i-1}}^{x_{i+1}} e^x \left(\frac{d\phi_i}{dx} + \phi_i\right)^2 dx + ge\underbrace{\phi_i^2(1)}_{=0} \quad i = 1,\dots,n-1$$

0.5 $$= \int_{x_{n-1}}^{x_n} e^x \left(\frac{d\phi_n}{dx} + \phi_n\right)^2 dx + ge\underbrace{\phi_n^2(1)}_{=1} \quad i = n$$

0.5 $$a(\phi_i, \phi_i) = \int_{x_{i-1}}^{x_i} e^x \left(\frac{1}{h} + \frac{x - x_i}{h}\right)^2 dx + \int_{x_i}^{x_{i-1}} e^x \left(\frac{-1}{h} - \frac{x - x_{i+1}}{h}\right)^2 dx$$

0.5 $$a(\phi_n, \phi_n) = \int_{x_{n-1}}^{x_n} e^x \left(\frac{1}{h} + \frac{x - x_n}{h}\right)^2 dx + ge$$